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The set of stable switching sequences for discrete-time linear switched systems ☆

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ABSTRACT

In this paper we study the characterization of the asymptotical stability for discrete-time switched linear systems. We first translate the system dynamics into a symbolic setting under the framework of symbolic topology. Then by using the ergodic measure theory, a lower bound estimate of Hausdorff dimension of the set of asymptotically stable sequences is obtained. We show that the Hausdorff dimension of the set of asymptotically stable switching sequences is positive if and only if the corresponding switched linear system has at least one asymptotically stable switching sequence. The obtained result reveals an underlying fundamental principle: a switched linear system either possesses uncountable numbers of asymptotically stable switching sequences or has none of them, provided that the switching is arbitrary. We also develop frequency and density indexes to identify those asymptotically stable switching sequences of the system.

1. Introduction

A switched system (continuous or discrete-time) consists of a family of subsystems and a rule that governs the dynamics switching among them. Transition sequences among subsystems are called switching sequences, which play a key role in determining the system behaviors. These types of models are found in many practical systems in which switching is necessary and essential as the system dynamics evolve. For instance, systems modeled for communication network dynamics, or for robot manipulators, or for traffic management can be characterized by switched systems. Furthermore, classical stochastic Markov jump systems can be viewed as switched systems associated with transition probability distributions.

Stability is always a fundamental issue for the study of dynamical systems since, once stability is guaranteed, system dynamics is predictable and thus desirable in applications. The stability analysis of switched systems is much more difficult and challenging than that of non-switched systems, including the linear case. The main challenge for the study of switched systems results from the fact that the switched mechanism basically is uncertain, although it may be subject to certain constraints. The stability consideration has to be taken into account for all possible switchings which leads to the difficulty for the analysis.

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Current study of stability of switched systems has been focused on the discussion of absolute stability. A switched system is said to be asymptotically stable (also called absolutely stable) if its all admissible trajectories converge to the origin. The existing approaches in literature for showing absolute stability of switched systems, to our best knowledge, are essentially based on the search of common Lyapunov functions [3,4,10,12,18] or variations of the same framework [2,7,14, 16,19,22], and the stability is characterized by the existence of positive definite solutions of a finite set of linear matrix inequalities (LMIs). Recent notable developments in this direction under the Lyapunov function framework include, among others, (i) to seek the largest set of switching sequences on which the system is asymptotically stable [13–15,17], and (ii) to determine the minimum dwell time such that a set of asymptotically stable switching sequences can be identified [5,6,21].

When the absolute stability (or other types of stability) of a switched system fails to exist, the system may still possess certain amount of asymptotically stable switching sequences. From the dynamical system point of view, a natural question is how to characterize those asymptotically stable switching sequences? More specifically, how many these switching sequence do we have? Or it is equivalent to ask what is the ‘size’ of the set which contains all asymptotically stable switching sequences. Furthermore, how to identify them?

In this paper we quantify those sets which contain asymptotically stable switching sequences by using ergodic measure theory. We first translate the switched system dynamics into a symbolic setting under the framework of symbolic topology. Then we study the ergodic measures defined on the compact symbolic space which contains all possible switching symbolic sequences, and define desirable ergodic measures to estimate the lower bound of the Hausdorff dimension of the set which contains asymptotically stable switching sequences for switched linear systems. We also develop frequency and density indexes to identify those asymptotically stable switching sequences of the system. Our approach is quite different from current literature and offers a new viewpoint for switched system dynamics.

The main contribution of the current paper are:

(1) By applying ergodic measure theory, we obtain lower bounds of the Hausdorff dimension of the set Σ_{stab} of asymptotically stable switching sequences provided that Σ_{stab} is not empty. We show that the Hausdorff dimension of Σ_{stab} is positive if and only if the corresponding switched linear system has at least one asymptotically stable switching sequence. This topological description reveals an underlying fundamental rule: a switched linear system either possesses an asymptotically stable switching sequence set with positive Hausdorff dimension or has none of them, provided that the switching is arbitrary.
(2) Although the question of numerical computation is not tackled in this paper, a lower bound of the Hausdorff dimension of Σ_{stab} reflects the minimum computational complexity (associated with a topological structure) for searching the entire set Σ_{stab}.
(3) By developing frequency and density indexes for switching sequences, we further establish a methodology to determine the elements in Σ_{stab}, which is different from current available literature.

Our main results provide an important guild line for quantitative existence of asymptotically stable switching sequences. Moreover the asymptotically stable switching sequences can be determined by examining frequency and density indexes of subsystem states.

The paper is organized as follows. Section 2 provides the problem formulation under the framework of symbolic topology. System dynamics is transformed into a compact symbolic space and definitions for asymptotically stability as well as almost sure stability are introduced. In Section 3, we introduce our recent results which will be needed for later discussion. The lower bound estimate in terms of Hausdorff dimension for the set of stable switching sequences is provided in Section 4. Section 5 discusses certain important characterizations of stable switching sequences. In particular, a classification result for systems consisting of two scalar subsystems is presented. The paper ends with concluding remarks.

Throughout the paper, we adopt the following notation: if $H \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, then $\|H\|$ denotes the corresponding matrix norm induced by a given vector norm $\|x\|$ of $\mathbb{R}^n$.

2. Topological formulation

In this paper we consider the following switched linear systems

$$x_{\ell+1} = H_{\omega_\ell} x_\ell, \quad \ell \geqslant 0, \tag{2.1}$$

where $x_\ell \in \mathbb{R}^n$ and $n \geqslant 2$ is fixed, ω_ℓ takes value in a finite set $\mathcal{A} = \{1, \ldots, \kappa\}$ with $\kappa \geqslant 2$, and $H_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{A}$. Let Σ_κ be the set of all possible mappings

$$\omega : \mathbb{Z}_+ \to \mathcal{A},$$

where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ denotes the nonnegative integer set. Each element in Σ_κ is called a *switching sequence* of system (2.1).

Now let us view Σ_κ as a symbolic sequence space with κ-number symbols, and define a distance function (or called a metric) as

$$\operatorname{dist}(\omega, \omega') = \rho^{-n(\omega,\omega')}, \quad \forall \omega, \omega' \in \Sigma_\kappa, \tag{2.2a}$$

where $\rho > 1$ is any prescribed constant and

$$n(\omega, \omega') = \inf\{\ell \in \mathbb{Z}_+ \cup \{0\} \mid \omega_\ell \neq \omega'_\ell\}. \tag{2.2b}$$

Then Σ_κ is a compact metric space. Next we introduce a forward transformation $\sigma : \Sigma_\kappa \to \Sigma_\kappa$ as

$$(\omega_0\omega_1\omega_2\cdots) \mapsto (\omega_1\omega_2\omega_3\cdots), \quad \forall \omega \in \Sigma_\kappa.$$

Note that σ is well defined since for any $(\omega_0\omega_1\cdots) \in \Sigma_\kappa$, we have $(\omega_1\omega_2\cdots) \in \Sigma_\kappa$ due to the definition of Σ_κ. In terms of topology terminology, σ is usually called one-sided shift of Σ_κ, and (Σ_κ, σ) is called one-sided full shift symbolic dynamical system. Let us denote

$$\mathcal{H} = \{H_1, \ldots, H_\kappa\}.$$

Then we use $(\mathcal{H}, \Sigma_\kappa)$ to represent the family of system (2.1) over all $\omega \in \Sigma_\kappa$ and we call $(\mathcal{H}, \Sigma_\kappa)$ an arbitrary switching system.

The formulation can further incorporate with cases in which switchings are subject to constraints induced by an irreducible $(0,1)$-matrix $A = [a_{ij}]$ of $\kappa \times \kappa$ with $a_{ij} = 0$ or 1. That is, if for $\omega_\ell, \omega_{\ell+1} \in \mathcal{A} = \{1, 2, \ldots, \kappa\}$, the pair symbol $\omega_\ell\omega_{\ell+1}$ appears in $\omega \in \Omega$ for any $\ell \in \mathbb{Z}_+$ then $a_{\omega_l\omega_{l+1}} = 1$, otherwise $a_{\omega_l\omega_{l+1}} = 0$. Thus this type of constraints cab be characterized by the following set

$$\Sigma_A = \{\omega = (\omega_0\omega_1\cdots\omega_\ell\omega_{\ell+1}\cdots) \in \Sigma_\kappa \mid a_{\omega_l\omega_{l+1}} = 1,\ l = 0, 1, \ldots\} \tag{2.3}$$

which is a subset of Σ_κ. We call the pair $(\mathcal{H}, \Sigma_A)$ the switch system subject to the transition matrix A. Denote by σ_A the σ restricted to the set Σ_A. It is easy to see that Σ_A is invariant under σ (i.e. $\sigma(\Sigma_A) \subset \Sigma_A$). We thus obtain a compact dynamical system (Σ_A, σ_A) which is a subsystem of (Σ_κ, σ) and is said to be *one-sided topological Markovian chain* with the transition matrix A.

Let A be a transition matrix and μ be a Borel probability measure defined on Σ_A. Then μ is called σ_A-invariant if $\mu(\sigma_A^{-1}B) = \mu(B)$ for any Borel subset B of Σ_A. μ is called to be ergodic if, in addition, $\mu(B) = 0$ or 1 whenever $\sigma_A^{-1}B = B$. We denote by $E(\Sigma_A, \sigma_A)$ the set of all ergodic measures on Σ_A. For a given one-sided topological Markov chain $(\boldsymbol{\Sigma}_A, \sigma_A)$ with a transition matrix A, one always can define an invariant ergodic measure μ under the one-sided shift σ_A according to the classical Krylov–Bogoliubov theorem [20,23].

Definition 1. Let Ω be a nonempty subset of Σ_κ and A be a transition matrix.

(a) A switching sequence $\omega = \{\omega_\ell\} \in \Omega$ is said to be *asymptotically stable* if

$$\lim_{\ell\to\infty} \|x_{\ell+1}\| = \lim_{\ell\to\infty} \|H_{\omega_\ell} x_\ell\| = 0 \tag{2.4}$$

for all $x_0 \in \mathbb{R}^n$.

(b) The switched linear system $(\mathcal{H}, \Omega)$ is said to be *asymptotically stable* if (2.4) holds for all $x_0 \in \mathbb{R}^n$ and $\omega \in \Omega$.

(c) Let $\mu \in E(\Sigma_A, \sigma_A)$. The switched linear system $(\mathcal{H}, \Sigma_A)$ is said to be *μ-almost sure stable* if there exists a subset Ω of Σ_A with $\mu(\Omega) = 1$ such that $(\mathcal{H}, \Omega)$ is asymptotically stable.

Remark 1. For switched linear systems, the asymptotically stable is equivalent to exponentially stable. Thus all conclusions for the asymptotical stability in this paper are also applied to exponential stability automatically.

Remark 2. Let $\omega = (\omega_0\omega_1\cdots) \in \Sigma_\kappa$. It is straightforward to see that the switched linear system with single element ω, $(\mathcal{H}, \{\omega\})$, is asymptotically stable if and only if there exists a positive integer k such that

$$\|H_{\omega_{k-1}} H_{\omega_{k-2}} \cdots H_{\omega_0}\| < 1.$$

3. Some preliminary results

For the continence of later discussion, let us give the definition Hausdorff dimension of a set in a metric space. Let X be a compact metric space with a metric $d(\cdot,\cdot)$. We denote

$$B(x, r) = \{y \in X \mid d(x, y) \leqslant r\}$$

which stands for a closed ball centered at $x \in X$ with radius $r > 0$. Recall that for $s \geqslant 0$ the s-Hausdorff measure of $Y \subseteq X$ for a given metric $d(\cdot,\cdot)$ is defined as

$$\mathscr{H}_d^s(Y) = \lim_{\delta\to 0} \inf\left\{\sum_i |A_i|^s : \bigcup_i A_i \supseteq Y \text{ and } \sup_i |A_i| < \delta\right\}$$

where $\{A_i\}$ is a countable cover of Y and $|A_i|$ represents the diameter of A_i in terms of $d(\cdot,\cdot)$. Furthermore, the *Hausdorff dimension* of Y is defined by

$$HD_d(Y) = \inf\{s \in \mathbb{R}_+ \colon \mathscr{H}_d^s(Y) = 0\}.$$

The following two lemmas are versions of Theorem 2 and Corollary 5, respectively, given in [9].

Lemma 1. *Let A be an irreducible transition matrix and $\mu \in E(\Sigma_A, \sigma_A)$. If the switched linear system $(\mathcal{H}, \Sigma_A)$ is μ-almost sure stable, then*

$$HDL_\rho(\Sigma_A) \geqslant HDL_\rho\big(\Sigma_{stab}(\mu; A)\big) \geqslant HDL_\rho(\mu) = \frac{h_\mu(\sigma_A)}{\ln \rho}, \tag{3.1}$$

where

$$\Sigma_{stab}(\mu; A) = \big\{\omega \in \Sigma_A \;\big|\; \big(\mathcal{H}, \{\omega\}\big) \text{ is asymptotically stable}\big\},$$

$HD_\rho(X)$ denotes the Hausdorff dimension of a set X in $\Sigma\kappa$, $HD_\rho(\mu)$ is the Hausdorff dimension of the measure μ and $h_\mu(\sigma_A)$ is the entropy of σ_A with respect to μ.

Let $p = (p_1, p_2, \ldots, p_\kappa)$ be a probability distribution on $\mathcal{A} = (1, 2, \ldots, \kappa)$ with $p_i > 0$, $i = 1, \ldots, \kappa$, and $\sum_{i=1}^{\kappa} p_i = 1$. Then the product measure μ defined by

$$\mu_p\big([i_0 \cdots i_l]\big) = p_{i_0} \cdots p_{i_l}, \tag{3.2}$$

where

$$[i_0 \cdots i_l] = \{\omega \in \Sigma_\kappa \mid \omega_0 = i_0, \ldots, \omega_l = i_l\}$$

is the *cylinder* defined by the word of length $\ell + 1$ for $(i_0 \cdots i_\ell) \in \mathcal{A}^{\ell+1}$ with any $\ell + 1 \in \mathbb{N}$. According to [20, Theorem 1.13] we know that

(1) μ_p has the full support. That is $supp(\mu_p) = \Sigma_\kappa$.
(2) μ_p is an ergodic σ_κ-invariant Borel probability measure on Σ_κ.

Furthermore, it is easy to see that μ_p is the natural *Markov measure* with respect to the transition probability matrix $P = \{p_{ij}\}$ with $p_{ij} = p_j$, $1 \leqslant i, j \leqslant \kappa$ and the stationary distribution p. According to Corollary 4 in [9], we have:

Lemma 2. *Consider the switched linear inclusion $(\mathcal{H}, \Sigma_\kappa)$. For any $t \in \mathbb{N}$, write*

$$\lambda_{i_0 i_1 \cdots i_{t-1}} = \|H_{i_{t-1}} \cdots H_{i_0}\|, \quad \forall (i_0 \cdots i_{t-1}) \in \mathcal{A}^t, \tag{3.3}$$

where $\|\cdot\|$ is the matrix norm. Let $p = (p_1, p_2, \ldots, p_\kappa)$ be a probability distribution on $\mathcal{A}$ and μ_p be the ergodic measure defined by (3.2)*. Then $(\mathcal{H}, \Sigma_\kappa)$ is μ_p-almost sure stable if and only if there is at least one $\hat{t} \in \mathbb{N}$ such that*

$$\prod_{(i_0 \cdots i_{\hat{t}-1}) \in \mathcal{A}^{\hat{t}}} \lambda_{i_0 i_1 \cdots i_{\hat{t}-1}}^{p_{i_0} p_{i_1} \cdots p_{i_{\hat{t}-1}}} < 1. \tag{3.4}$$

Let the ergodic invariant measure μ_p be defined by (3.2). Then the corresponding measure entropy is given by [20, p. 102]

$$h_{\mu_p}(\sigma) = -\sum_{i=1}^{\kappa} p_i \ln p_i. \tag{3.5}$$

Furthermore, with the distance defined as (2.2), the Hausdorff dimension of Σ_κ is obtained as

$$HD_\rho(\Sigma_\kappa) = \frac{\ln \kappa}{\ln \rho}.$$

From Lemmas 1, 2, we have:

Corollary 1. *Under the assumptions of Lemma* 2*, we have*

$$\frac{\ln \kappa}{\ln \rho} \geqslant HD_\rho(\Sigma_{stab}) \geqslant \frac{-\sum_{i=1}^{\kappa} p_i \ln p_i}{\ln \rho}. \tag{3.6}$$

4. Topological characterization of stable switching sequences

Let us consider the switched linear system $(\mathcal{H}, \Sigma_\kappa)$ with $\kappa \geqslant 2$ defined in Section 2, and denote

$$\Sigma_{stab} = \{\omega \in \Sigma_\kappa, \text{ where the switching sequence } \omega \text{ is asymptotically stable}\},$$

which is the set of all stable switching sequences in Σ_κ. In this section we shall study the topological characterization of Σ_{stab}.

Theorem 1. *Consider the switched linear system $(\mathcal{H}, \Sigma_\kappa)$. If there exists $H_{i_0} \in \mathcal{H}$ such that $\|H_{i_0}\| < 1$, then there exists $\alpha \in (0,1)$ such that the Hausdorff dimension of $HD_\rho(\Sigma_{stab})$ satisfies the following inequality*:

$$HD_\rho(\Sigma_{stab}) \geqslant \frac{-\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha) + (1-\alpha)\ln(\kappa-1)}{\ln \rho} > 0. \tag{4.1}$$

Proof. Since $\|H_{i_0}\| < 1$ and κ is finite, there always exists $\alpha \in (0,1)$ such that the following estimation:

$$\|H_{i_0}\|^\alpha \left(\prod_{i=1, i \neq i_0}^{\kappa} \|H_i\| \right)^{\frac{1-\alpha}{\kappa-1}} < 1 \tag{4.2}$$

holds. Let $p = (p_1, \dots p_\kappa)$ be a probability distribution on $\mathcal{A}$ with $p_{i_0} = \alpha$, $p_i = \frac{1-\alpha}{\kappa-1}$, $\forall 1 \leqslant i \leqslant \kappa$, $i \neq i_0$. Similar to the measure introduced by (3.2), we now define an ergodic σ_κ-invariant Borel probability measure on Σ_κ as follows:

$$\mu_p\big([i_0 \cdots i_l]\big) = p_{i_0} \cdots p_{i_l}, \tag{4.3}$$

where

$$[i_0 \cdots i_l] = \{\omega \in \Sigma_\kappa \mid \omega_0 = i_0, \dots, \omega_l = i_l\}.$$

The inequality (4.2) implies that the stable condition (3.4) holds for $\hat{t} = 1$. Thus according to Lemma 2, we know that the switched linear system $(\mathcal{H}, \Sigma_\kappa)$ is μ_p-almost sure stable. Notice that in this case we have

$$-\sum_{i=1}^{\kappa} p_i \ln p_i = -\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha) + (1-\alpha)\ln(\kappa-1).$$

Therefore (4.1) follows from Corollary 1. □

Theorem 1 can be further generalized into the following:

Theorem 2. *Consider the switched linear system $(\mathcal{H}, \Sigma_\kappa)$. Suppose that there exists a stable subsystem H_{i_0} in the switched system $(\mathcal{H}, \Sigma_\kappa)$. That is, there exists $i_o \in \mathcal{A}$ such that the system*

$$x_{l+1} = H_{i_0} x_l$$

is asymptotically stable. Then there exists $\alpha \in (0,1)$ such that the lower bound of Hausdorff dimension of Σ_{stab} satisfies (4.1).

Proof. Notice that the subsystem with the matrix H_{i_0} is asymptotically stable, thus there exists a positive integer $\hat{t}_0$ such that

$$\|\overbrace{H_{i_0} H_{i_0} \cdots H_{i_0}}^{\hat{t}_0 \text{ times}}\| < 1.$$

For any $\alpha \in (0,1)$, let us choose a probability distribution $p = (p_1, \dots p_\kappa)$ with $p_{i_0} = \alpha$, $p_i = \frac{1-\alpha}{\kappa-1}$, $\forall 1 \leqslant i \leqslant \kappa$, $i \neq i_0$ on $\mathcal{A}$. Notice that there always exists some $\alpha \in (0,1)$ such that

$$\|H_{i_0} H_{i_0} \cdots H_{i_0}\|^{\alpha^{\hat{t}_0}} \prod_{(j_0 \cdots j_{\hat{t}_0 - 1}) \in \mathcal{A}^{\hat{t}_0} \setminus \{i_0 \cdots i_0\}} \|H_{j_{\hat{t}_0-1}} \cdots H_{j_0}\|^{p_{j_0} p_{j_1} \cdots p_{j_{\hat{t}_0-1}}} < 1, \tag{4.4}$$

since $p_{j_0} p_{j_1} \cdots p_{j_{\hat{t}_0-1}} \to 0$ as $\alpha \to 1$ because it contains the term $(1-\alpha)$. If we construct an ergodic σ_κ-invariant Borel probability measure μ_p on Σ_κ as one presented in Theorem 1, then (4.4) implies that $(\mathcal{H}, \Sigma_\kappa)$ is μ_p-almost sure stable, which yield (4.1) by applying Corollary 1 again. □

Remark 3. Let us denote by $h(\alpha)$ the right-hand side of Eq. (4.1). As a function of $\alpha \in (0,1)$, $h(\alpha)$ is increasing in the interval $(0, \frac{1}{\kappa})$ and is decreasing in $(\frac{1}{\kappa}, 1)$. The maximum value attains at $\alpha = \frac{1}{\kappa}$:

$$h\left(\frac{1}{\kappa}\right) = \frac{\ln \kappa}{\ln \rho},$$

which is the Hausdorff dimension of the whole symbolic space Σ_κ associated with the given metric (2.2).

Thus, we have

Corollary 2. *Consider the switched linear system $(\mathcal{H}, \Sigma_\kappa)$.*

(1) *If there exists a positive integer $\hat{t}_0$ such that*

$$\prod_{(j_0 \cdots j_{\hat{t}_0 - 1}) \in \mathcal{A}^{\hat{t}_0}} \|H_{j_{\hat{t}_0-1}} \cdots H_{j_0}\| < 1,$$

then $HD_\rho(\Sigma_{stab}) = \frac{\ln \kappa}{\ln \rho}$, which is the Hausdorff dimension of the whole space Σ_κ.

(2) *In particular, if $\|H_i\| \leqslant 1$ for all $i \in \mathcal{A}$ and there is at least j with $1 \leqslant j \leqslant \kappa$ such that $\|H_j\| < 1$, then $HD_\rho(\Sigma_{stab}) = \frac{\ln \kappa}{\ln \rho}$.*

Proof. For (1), since the assumption implies the inequality (4.4) holds, the result follows from Theorem 2 by taking $\alpha = \frac{1}{\kappa}$.

For (2), the conclusion follows from (1) by taking $\hat{t}_0 = 1$. $\square$

Next we turn our attention to the case in which there is an asymptotically stable switching sequence in the switched system $(\mathcal{H}, \Sigma_\kappa)$. We will again show that the Hausdorff dimension of Σ_{stab} is also positive, but the approach is somewhat different since there may not exist a probability distribution on $\mathcal{A}$ as those presented in Theorems 1 and 2. Here we provide an example to illustrate the issue.

Example 1. We consider a switched system with two subsystems. Let $\mathcal{A} = \{1,2\}$ and

$$H_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then we have $\|H_2 H_1\| = 0.6708 < 1$ and so the switched system $(\mathcal{H}, \Sigma_2)$ has an asymptotically stable sequence $\omega = (212\,121\cdots)$. But for any probability distribution $p = (\alpha, 1-\alpha)$ on $\mathcal{A} = \{1,2\}$ with $\alpha \in (0,1)$, we have

$$\|H_1 H_1\|^{\alpha^2} \|H_1 H_2\|^{\alpha(1-\alpha)} \|H_2 H_1\|^{(1-\alpha)\alpha} \|H_2 H_2\|^{(1-\alpha)^2} > 1.$$

This implies that the condition (4.4) does not hold, and thus we cannot directly follow the previous approach. Note that even though both H_1 and H_2 are unstable, our next theorem indicates that the system has infinitely many stable switching sequences.

In order to overcome the problem mentioned above, we introduce a so called m-lifting system for our approach. Suppose that $(\mathcal{H}, \Sigma_\kappa)$ has an asymptotically stable switching sequence $\omega = (\omega_0 \omega_1, \ldots) \in \Sigma_\kappa$. According to Remark 2, we know that there exists a positive integer m with $\omega_{m-1}, \ldots, \omega_0$ in $\mathcal{A}$ such that

$$\|H_{\omega_{m-1}} \cdots H_{\omega_0}\| < 1. \tag{4.5}$$

We now define

$$\tilde{\mathcal{H}}_m = \left\{ H_{j_{m-1}} H_{j_{m-2}} \cdots H_{j_0} \,\middle|\, (j_{m-1}, \ldots, j_0) \in \mathcal{A}^m \right\}.$$

Let us consider the switched system $(\tilde{\mathcal{H}}_m, \Sigma_{\kappa^m})$, which consists of κ^m numbers subsystems, each associated with matrix $H_{j_{m-1}} H_{j_{m-2}} \cdots H_{j_0}$. More precisely, we consider the switched system:

$$\bar{x}_{\ell+1} = (H_{j_{m-1}} \cdot H_{j_{m-2}} \cdots H_{j_0}) \bar{x}_\ell,$$

for any $j_s \in \mathcal{A}$ for $0 \leqslant s \leqslant m-1$. Clearly, $\Sigma_{\kappa^m} = \Sigma_\kappa$ according to the definition of symbolic space Σ_κ given in Section 2. We next claim that the Hausdorff dimension of the set of stable switching sequences in Σ_{κ^m} is positive. According to (4.5), we always can choose an $\alpha \in (0,1)$ such that

$$\|H_{\omega_{m-1} \cdots \omega_0}\|^\alpha \left(\prod_{(j_{m-1} \cdots j_0) \in \mathcal{A}^m \setminus (\omega_{m-1} \cdots \omega_0)} \|H_{j_{m-1} \cdots j_0}\| \right)^{\frac{1-\alpha}{\kappa^m - 1}} < 1. \tag{4.6}$$

Similar to the proof of Theorem 1, we introduce a block probability distribution $p = (p_1, \ldots, p_{\kappa^m})$ on $\mathcal{A}^m$ with $p(\omega_0\omega_1\cdots\omega_{m-1}) = \alpha$ and $p(\omega_{j_0}\omega_{j_1}\cdots\omega_{j_{m-1}}) = \frac{1-\alpha}{\kappa^m-1}$ for any $\omega_{j_0}\omega_{j_1}\cdots\omega_{j_{m-1}} \neq \omega_0\omega_1\cdots\omega_{m-1}$. We denote by μ_p the product measure generated by p_i. Note that in this case μ_p is a σ_κ^m-ergodic measure. σ_κ^m is actually the one-sided shift σ_{κ^m} in Σ_{κ^m}. Since the distance defined in (2.2a) is a dimension metric *w.r.t.* $(\Sigma_\kappa, \sigma_\kappa^m)$ with a skewness ρ^m, it follows from Theorem 1 in [8] that

$$HD_\rho(\mu_p) = \frac{h_{\mu_p}(\sigma_\kappa^m)}{\ln\rho^m} = \frac{-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha) + (1-\alpha)\ln(\kappa^m - 1)}{m\ln\rho}.$$

Now according to Theorem 1, the switched system $(\tilde{\mathcal{H}}_m, \Sigma_{\kappa^m})$ is μ_p almost sure stable, and by Lemma 1 the Hausdorff dimension of the set of stable switching sequences in Σ_{κ^m} is no smaller than

$$\frac{-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha) + (1-\alpha)\ln(\kappa^m - 1)}{m\ln\rho}. \tag{4.7}$$

Thus, we thus have the following result:

Theorem 3. *If the switched system $(\mathcal{H}, \Sigma_\kappa)$ has an asymptotically stable switching sequence, then the Hausdorff dimension of the set of stable switching sequences is positive. More precisely, there exists $\alpha \in (0,1)$ such that*

$$HD_\rho(\Sigma_{stab}) \geqslant \frac{-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha) + (1-\alpha)\ln(\kappa^m - 1)}{m\ln\rho}, \tag{4.8}$$

where m is determined by (4.6).

Remark 4. We here give four important remarks:

(1) Theorem 3 contains those results given by Theorems 1 and 2. In other words, Theorems 1 and 2 are special cases of Theorem 3.
(2) From Theorems 2 and 3, the obtained lower bound of the Hausdorff dimension of Σ_{stab} when the system has an asymptotically stable subsystem is larger than that when the system has an asymptotically stable switching sequence.
(3) Theorem 3 also indicates that the Hausdorff dimension of the set of asymptotically stable switching sequences is positive if and only if the switched system $(\mathcal{H}, \Sigma_\kappa)$ has at least one asymptotically stable switching sequence.
(4) It is known that if a set in a metric space has positive Hausdorff dimension, then it must contain uncountable many elements. Therefore, Theorem 3 implies that if a switched linear system has an asymptotically stable switching sequence, then it possesses uncountable many stable switching sequences.

We conclude this section with following example:

Example 2. Consider the switched system $(\mathcal{H}, \Sigma_\kappa)$ with $\kappa = 2$ and $\mathcal{H} = \{H_1, H_2\}$, where

$$H_1 = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.2 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.9 & 0.4 \\ 0.5 & 0.2 \end{bmatrix}.$$

It is obvious that this system is not uniformly stable since the spectral radius of H_2 is bigger than 1. Since

$$\lambda_{11} = \|H_1H_1\| = 0.4040, \qquad \lambda_{12} = \|H_1H_2\| = 0.7432,$$
$$\lambda_{21} = \|H_2H_1\| = 1.1377, \qquad \lambda_{22} = \|H_2H_2\| = 1.2545,$$
$$\prod_{(i_0i_1)\in\{1,2\}^2} \lambda_{(i_0i_1)} = 0.4285 < 1.$$

It follows from Corollary 2 that then $HD_\rho(\Sigma_{stab}) = \frac{\ln 2}{\ln\rho}$, which is the Hausdorff dimension of the whole space Σ_2.

5. Methods for searching stable switching sequences

In previous section, we show that if a switched linear system has an asymptotically stable switching sequence then it must have uncountable many asymptotically stable switching sequences. This offers a fundamental indication which implies the existence of a great degree of freedom for the design of asymptotically switching sequences for the underlying system. A further question is how to identify those stable switching sequences, which is critical and essential since, in most applications, only stable switching sequences are desirable and can function in expected dynamical behaviors. In this section, we devote our effort to identify those asymptotically switching sequences. In order to pursue a further discussion, we need to introduce some notations.

For a given $i \in \mathbb{N}$, let $\chi_i(\cdot)$ be the characteristic function defined on $\mathbb{N}$, that is,

$$\chi_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Consider the switched system $(\mathcal{H}, \Sigma_\kappa)$. For a given switching sequence $\omega = (\omega_0\omega_1\cdots) \in \Sigma_\kappa$, we define

$$S_k^i(\omega) = \sum_{l=0}^{k-1} \chi_i(\omega_l), \quad i = 1, \ldots, \kappa,$$

which represents the cardinality of i occurred in the finite sequence $(\omega_0 \cdots \omega_{k-1})$, and

$$\underline{D}^i(\omega) = \liminf_{k\to\infty} \frac{S_k^i(\omega)}{k} \tag{5.1}$$

which stands for the lower density of the symbol "i" appearing in ω, $i = 1, \ldots, \omega$. Finally, for $i = 1, \ldots, \kappa$ and $\alpha \in [0, 1]$, we denote

$$S_\alpha^i = \left\{\omega \in \Sigma_\kappa \;\middle|\; \underline{D}^i(\omega) \geqslant \alpha\right\} \tag{5.2}$$

the set of all switching sequences in which the lower density of i appears is not less than α.

Theorem 4. *Consider the switched system $(\mathcal{H}, \Sigma_\kappa)$. Suppose that there exists at least an $i_0 \in \mathcal{A}$ such that $\|H_{i_0}\| < 1$. Then for those $\alpha \in [0, 1]$ satisfying*

$$a_{i_0}^{1-\alpha} \|H_{i_0}\|^\alpha < 1, \qquad a_{i_0} = \max\left\{\|H_j\|,\ j \in \mathcal{A},\ j \neq i_0\right\}, \tag{5.3}$$

we have

$$\Sigma_{stab} \supset S_\alpha^{i_0}. \tag{5.4}$$

Proof. If $a_{i_0} < 1$, then it is straightforward to see that all switching sequences in Σ_κ are the system. That is, $\Sigma_{stab} = \Sigma_\kappa$. Thus (5.4) is obviously true.

If $a_{i_0} = 1$, then according to the part (2) of Corollary 2 we know that (5.4) holds too.

Now we consider the case for $a_{i_0} > 1$. According to (5.3), for a given $0 < q < 1$ we choose $\varepsilon_0 > 0$ to be small enough such that

$$a_{i_0}^{1-\alpha-\varepsilon_0} \|H_{i_0}\|^{(\alpha-\varepsilon_0)} \leqslant q < 1. \tag{5.5}$$

For each $\omega \in S_\alpha^{i_0}$, it suffices to show that it is asymptotically stable. According to the definition of $S_\alpha^{i_0}$, we know that there exists a positive integer l_0 such that

$$\frac{S_l^{i_0}(\omega)}{l} \geqslant \alpha - \varepsilon_0, \quad \forall l \geqslant l_0. \tag{5.6}$$

Noticing (5.3), (5.5) and (5.6), we have

$$\begin{aligned} \|x_l\| &= \|H_{\omega_{l-1}} \cdots H_{\omega_0} x_0\| \leqslant \|H_{\omega_{l-1}}\| \cdots \|H_{\omega_0}\| \|x_0\| \\ &\leqslant \|H_{i_0}\|^{S_l^{i_0}(\omega)} a_{i_0}^{l-S_l^{i_0}(\omega)} \|x_0\| = \left[\|H_{i_0}\|^{\frac{S_l^{i_0}(\omega)}{l}} a_{i_0}^{1-\frac{S_l^{i_0}(\omega)}{l}}\right]^l \|x_0\| \\ &\leqslant \left[\|H_{i_0}\|^{\alpha-\varepsilon_0} a_{i_0}^{1-\alpha-\varepsilon_0}\right]^l \|x_0\| \leqslant q^l \|x_0\|, \end{aligned}$$

for any $l \geqslant l_0$. The third inequality above comes from the fact that the function $f(z) = \|H_{i_0}\|^z a_{i_0}^{1-z}$ decreases as z increases since $\|H_{i_0}\| < 1$ and $a_{i_0} > 1$. Thus the switching sequence ω is an asymptotically stable, which yields $\omega \in \Sigma_{stab}$. $\square$

Now we consider that case in which the switched system $(\mathcal{H}, \Sigma_\kappa)$ has an asymptotically stable switching sequence. We know that there exists a finite matrix sequence $H_{i_{m-1}}, \ldots, H_{i_0} \in \mathcal{H}$ such that

$$\|H_{i_{m-1}} \cdots H_{i_0}\| < 1. \tag{5.7}$$

Similar to the proof of Theorem 3, we introduce an m-lifting system $(\tilde{\mathcal{H}}_m, \Sigma_{\kappa^m})$, and let $\tilde{i} = (i_{m-1} \cdots i_0) \in \mathcal{A}^m$. For $\tilde{\omega} = (\tilde{\omega}_0\tilde{\omega}_1\tilde{\omega}_2\cdots) \in \Sigma_{\kappa^m}$ with $\tilde{\omega}_j \in \mathcal{A}^m$, we define

$$S_k^{\tilde{i}}(\tilde{\omega}) = \sum_{l=0}^{k-1} \chi_{\tilde{i}}(\tilde{\omega}_l), \qquad \underline{D}^{\tilde{i}}(\tilde{\omega}) = \liminf_{k\to\infty} \frac{S_k^{\tilde{i}}(\tilde{\omega})}{k},$$

and for $\alpha \in [0,1]$,

$$S_\alpha^{\tilde{i}} = \{\tilde{\omega} \in \Sigma_{\kappa^m} \mid \underline{D}^{\tilde{i}}(\tilde{\omega}) \geqslant \alpha\}.$$

By repeating the proof of Theorem 4, we have:

Theorem 5. *Suppose that the switched system $(\mathcal{H}, \Sigma_\kappa)$ has an asymptotically stable sequence. Then for those $\alpha \in [0,1]$ satisfying*

$$a_{i_0}^{1-\alpha} \|H_{i_{m-1}} \cdots H_{i_0}\|^\alpha < 1, \tag{5.8}$$

$$a_{i_0} = \max_{(j_{m-1}\cdots j_0)\in\mathcal{A}^m,\, (j_{m-1}\cdots j_0)\neq(i_{m-1}\cdots i_0)} \{\|H_{j_{m-1}} \cdots H_{j_0}\|\}, \tag{5.9}$$

we have

$$\Sigma_{stab} \supset S_\alpha^{\tilde{i}}. \tag{5.10}$$

Example 3. Let us revisit the first example, where $\mathcal{A} = \{1,2\}$ and

$$H_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

Notice that both subsystems H_1 and H_2 are not asymptotically stable. For simplicity, we choose $m = 2$. Let $\tilde{i} \in \{1,2\} \times \{1,2\} = \mathcal{A}^2$. Notice in this case

$$a_{i_0} = \max_{(j_1 j_0)\in\mathcal{A}^2,\, (j_1 j_0)\neq(21)} \{\|H_{j_1} H_{j_0}\|\} = 16.$$

If we let $\alpha = \frac{7}{8}$, then we have

$$a_{i_0}^{1-\alpha} \|H_2 H_1\|^\alpha = 0.9972 < 1.$$

This implies that

$$S_\alpha^{\tilde{i}} = \left\{\tilde{\omega} \in \Sigma_{2^2} \;\middle|\; \underline{D}^{\tilde{i}}(\tilde{\omega}) \geqslant \frac{7}{8}\right\}$$

is a set of asymptotically stable sequences with the lower density of the pair ‘21’ being no less than $\frac{7}{8}$. In other words, for any switching sequence $\tilde{\omega} = (\tilde{\omega}_0\tilde{\omega}_1 \ldots) \in \Sigma_{2^2}$, if the eventual appearance frequency of ‘21’ is at least $\frac{7}{8}$, then this switching sequence is asymptotically stable. One can obtain many asymptotically stable sequences by this way. For instance, notice that

$$\|H_2 H_1 H_1\| = 0.2012,$$

and

$$a_{i_0} = \max_{(j_2 j_1 j_0)\in\mathcal{A}^3,\, (j_2 j_1 j_0)\neq(211)} \{\|H_{j_2} H_{j_1} H_{j_0}\|\} = 64.$$

If we let $\alpha = \frac{6}{8}$, then we have

$$a_{i_0}^{1-\alpha} \|H_2 H_1 H_1\|^\alpha = 0.8497 < 1.$$

This means that for any switching sequence $\tilde{\omega} = (\tilde{\omega}_0\tilde{\omega}_1 \ldots) \in \Sigma_{2^3}$, if the eventual appearance frequency of ‘211’ is at least $\frac{6}{8}$, then it is asymptotically stable. Furthermore, according to Theorem 3, we know that the Hausdorff dimension of Σ_{stab} (where $\rho = 2$) is given by

$$HD_\rho(\Sigma_{stab}) \geqslant 0.5044.$$

So far in this paper, all our obtained results are based on the lower estimates of the Hausdorff dimension of the set of asymptotically stable switching sequences Σ_{stab}. Obviously, if we could obtain the exact Hausdorff dimension instead of lower bound, we would be able to say more about the topological description of Σ_{stab}. Unfortunately, it is often difficult to determine the Hausdorff dimension of a set in a metric space. Next we will present a case to demonstrate a way how to compute the Hausdorff dimension for certain sets containing asymptotically stable switching sequences.

Suppose that the switched linear system $(\mathcal{H}, \Sigma_\kappa)$ has an asymptotically stable switching sequence. Then there exists a switching sequence $(i_0 i_1 \cdots) \in \Sigma_\kappa$ such that

$$\|H_{i_{l-1}} \cdots H_{i_0}\| \to 0, \quad \text{as } l \to \infty. \tag{5.11}$$

Let us denote

$$a = \max_{i \in \mathcal{A}} \{\|H_i\|\}.$$

According to (5.11), there exists an integer $t_0 \geqslant 0$ such that

$$a\|H_{i_{t_0-1}} \cdots H_{i_0}\| < 1. \tag{5.12}$$

Let $\tilde{\omega}_j = (i_{t_0-1} \cdots i_0 j)$, $j = 1, \ldots, \kappa$. Define a subset of Σ_κ as

$$\Omega = \{(\tilde{j}_0 \tilde{j}_1 \cdots) \mid \tilde{j}_l \in \{\tilde{\omega}_1, \ldots, \tilde{\omega}_\kappa\}\}. \tag{5.13}$$

It is clear from (5.13) that the switched linear system $(\mathcal{H}, \Omega)$ is asymptotically stable. The Hausdorff dimension of Ω is given by the following theorem:

Theorem 6. *As a subset of Σ_κ, the Hausdorff dimension of Ω is*

$$HD_\rho(\Omega) = \frac{\ln \kappa}{(t_0 + 1) \ln \rho}.$$

Proof. Recall that the distance $dist(\omega, \omega')$ between two elements ω, ω' of $\sum_\kappa$ is defined to be $\rho^{-n(\omega,\omega')}$, where $n(\omega, \omega') = \inf\{l \in \mathbb{Z} \cup \{0\} \mid \omega_l \neq \omega'_l\}$.

Consider the following map $f_i : \Omega \to \Omega$ for each $\tilde{\omega}_i$ with $i = 1, 2, \ldots, \kappa$:

$$f_i(\tilde{j}_0 \tilde{j}_1 \cdots) = (\tilde{\omega}_i \tilde{j}_0 \tilde{j}_1 \cdots).$$

Then, it is easy to check that every f_i is a similitude with contraction ratio $\rho^{-(t_0+1)}$ such that $f_{i_1}(\Omega) \cap f_{i_2}(\Omega) = \emptyset$ for all $i_1 \neq i_2$. This means that Ω is a self-similar set satisfying the open set condition [11, pp. 129–130]. Therefore, its Hausdorff dimension $HD_\rho(\Omega)$ is the unique solution s to the equation $\kappa(\rho^{-(t_0+1)})^s = 1$, that is,

$$HD_\rho(\Omega) = \frac{\ln \kappa}{(t_0 + 1) \ln \rho}. \qquad \square$$

Remark 5. Theorem 6 may be used as alternative arguments for Theorems 1–3 if we only focus on showing the Hausdorff dimension to be positive. However, the lower bounds provided in Theorems 1–3 provide more information which allow us to compare with these estimates between different switched systems.

For a switched linear system consisting of two subsystem, we have more detail result. In this case we follow the convention and set $\rho = 2$ in (2.2) so that $HD_\rho(\Sigma_2) = 1$.

Theorem 7. *Consider the switched linear system $(\mathcal{H}, \Sigma_2)$ with $\mathcal{H} = \{H_1, H_2\}$. If $\|H_1\| < 1$, then*

$$\Sigma_{stab} \supset S^1_\alpha,$$

where S^1_α is defined by (5.2) and $\alpha \in [\frac{1}{2}, 1]$ satisfies the condition (5.3) with $i_0 = 1$. Moreover, we obtain

$$HD_2(\Sigma_{stab}) = \frac{-\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha)}{\ln 2} = HD_2(S^1_\alpha).$$

Proof. The first part is a special case of Theorem 4 with $\kappa = 2$. For the second part, the first equality results from Theorem 1 with $\kappa = 1$. The second equality is derived by following lemma by Besicovitch [1]. $\square$

Lemma 3. *Consider the symbolic space Σ_2. Let S^i_α, $i=1,2$, be defined as (5.2), where $\alpha\in[0,1]$. Then*

$$HD_2\big(S^i_\alpha\big)=\begin{cases}\frac{-\alpha\ln\alpha-(1-\alpha)\ln(1-\alpha)}{\ln 2}, & \alpha\geqslant\frac{1}{2},\\ 1, & \alpha<\frac{1}{2}.\end{cases}$$

Example 4. Consider the switched system $(\mathcal{H},\Sigma_2)$ with $\mathcal{H}=\{H_1,H_2\}$, where

$$H_1=\begin{bmatrix}2.0 & -1.25\\ 3.0 & -2.0\end{bmatrix},\qquad H_2=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Since H_1 has eigenvalues $\lambda=\pm 0.5$, its corresponding subsystem is asymptotically stable. But it is not absolutely stable since H_2 has an eigenvalue 1 and H_2H_1 has an eigenvalue 2. To estimate the Hausdorff dimension of Σ_{stab}, a feasible way is to define a new matrix norm such that the norm of H_1 is less than 1. Since

$$P^{-1}H_1P=J=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},$$

where

$$P=\begin{bmatrix}5 & 1\\ 6 & 2\end{bmatrix},\qquad P^{-1}=\frac{1}{4}\begin{bmatrix}2 & -1\\ -6 & 5\end{bmatrix}.$$

For $z\in\mathbb{R}^2$, define a new norm

$$\|z\|_P=\big\|P^{-1}z\big\|_2,$$

where $\|\cdot\|_2$ is the Euclidean norm. A direct computation shows that

$$\|H_1\|_P\leqslant 0.5,\qquad \|H_2\|_P=\frac{1}{4}\sqrt{1040}<8.07.$$

Thus, if $\alpha\geqslant 0.7508$, we have

$$\|H_1\|_P^\alpha\|H_2\|_P^{1-\alpha}<1.$$

Therefore, from Theorem 1 by taking $\rho=2$, we have

$$HD_\rho(\Sigma_{stab})=\frac{-\alpha\ln\alpha-(1-\alpha)\ln(1-\alpha)}{\ln\rho}\simeq 0.81.$$

More particularly, for switched linear scalar systems with two subsystems, we have the following classification:

Corollary 3. *Let $\alpha\in[0,1]$ and μ_α be the product measure on Σ_2 with respect to the probability distribution $(\alpha,1-\alpha)$ on $\{1,2\}$. For $\mathcal{H}=\{a_1,a_2\}$, we consider the scalar switched linear system $(\mathcal{H},\Sigma_2)$.*

(1) *If $|a_1|\geqslant 1$ as well as $|a_2|\geqslant 1$, then $(\mathcal{S},\Sigma_2)$ is complete unstable.*
(2) *If either $|a_1|<1$ or $|a_2|<1$, then, for each α which satisfies*

$$|a_1|^\alpha|a_2|^{1-\alpha}<1,$$

$(\mathcal{H},\Sigma_2)$ is μ_α-almost sure stable and the set of all stabilizing switching sequences has a positive dimension. Precisely,

$$1\geqslant HD_2(\Sigma_{stab})\geqslant\begin{cases}\frac{-\alpha\ln\alpha-(1-\alpha)\ln(1-\alpha)}{\log 2} & \text{if } \alpha\geqslant\frac{1}{2},\\ 1 & \text{if } \alpha<\frac{1}{2}.\end{cases}$$

And

$$\Sigma_{stab}\supset S^1_\alpha\quad\text{or}\quad\Sigma_{stab}\supset S^2_\alpha.$$

(3) *If $|a_1a_2|<1$, then $(\mathcal{H},\Sigma_2)$ is $\mu_{1/2}$-almost sure stable and*

$$HD_2(\Sigma_{stab})=1.$$

(4) *If $|a_1|<1$ as well as $|a_2|<1$, then $(\mathcal{H},\Sigma_2)$ is asymptotically stable.*

6. Concluding remarks

By viewing switching sequences as the elements in symbolic space and applying ergodic measure theory, we show a fundamental characteristic for switched linear systems: a switched linear system either possesses asymptotically stable switching sequence set with positive Hausdorff dimension or has none of them, provided the switching is arbitrary. The outcome provides a necessary base for the search of asymptotically stable switching sequences for switched linear systems. It indicates that there is a great degree of freedom for those asymptotically stable switching sequences and thus the design of switching rules has many flexibilities.

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